Uncertainty in the measurement of low-frequency magnetic fields — Rough assessment of the error due to the finite size of the sensor in a non-uniform field

Daniele Andreuccetti, CNR-IFAC, Florence

There are, as is well known, limitations to the possibility of performing accurate measurements of electric, magnetic and electromagnetic fields at points located very close to the source. These limitations, which often translate into a larger measurement uncertainty, depend on several factors, including:

- the size of the sensor (meaning, by this term, the sensitive part of the measuring probe), since it is physically impossible to bring the electrical centre of the sensor closer to the source structures beyond a certain limit;
- the possibility that the direct coupling between sensor and source alters the distribution of charges and/or currents on the structures of the latter, modifying its emission characteristics;
- the rate of variation of the field strength as a function of the distance from the source, because of which the field can be markedly non-uniform within the portion of space occupied by the sensor.

In this technical note we want to analyse the last of the factors just listed, i.e. to investigate the measurement error due to the size of the sensor when it is immersed in a strongly non-uniform field. We shall consider, in particular, the measurement of the magnetic field under quasi-static conditions, referring to the following scenario (see also Figure 1), simple enough to be handled analytically:

- a magnetic field source with cylindrical symmetry in a sinusoidal quasi-static regime, geometrically represented by an indefinitely long straight conductor;
- a single-loop sensor of square shape with side 2L, lying on a plane that also contains the source conductor and aligned with two sides parallel to it; the centre of the loop is located at a distance d from the conductor itself.

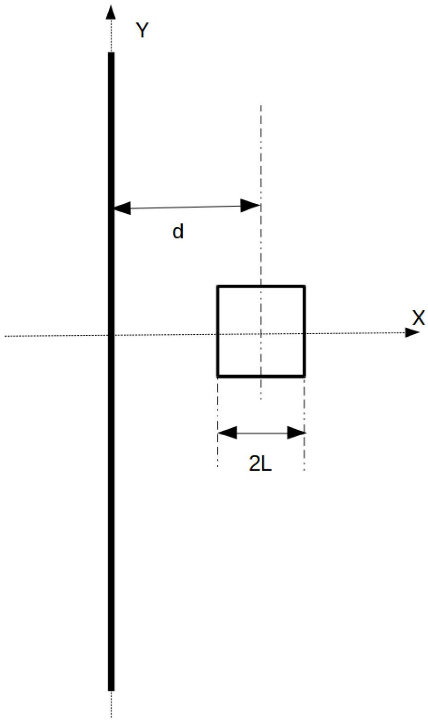

Figure 1. *Measurement scenario: an indefinitely long straight source conductor along the Y axis, and a square single-turn sensor of side 2L whose centre lies at a distance d from the conductor, on the same plane.*

This scenario — however extremely simplified — can provide interesting information on the error we are looking for (at least for the purpose of formulating a rough recommendation), also because it can be argued that some of the simplifications present in the model (square shape of the sensor and alignment with the source) go in the direction of providing an approximate overestimate of the error itself.

As regards the behaviour of the field strength as a function of the distance from the source conductor, we shall refer to three particular cases.

Case 1 — Magnetic flux density decaying as 1/r away from the source conductor. The first case corresponds to the Biot–Savart field and is, strictly speaking, the only physically possible one when dealing with an indefinitely long single-wire straight conductor; on the other hand, this case alone is not sufficient to exhaust the subject, because in real situations one almost always deals with fields that decay more rapidly and therefore give rise to a larger error than the one that can be estimated under the 1/r assumption.

Case 2 — Magnetic flux density decaying as $1/r^2$ away from the source conductor. In the second case we shall assume a field that decays inversely with the square of the distance away from the source conductor, while still keeping the cylindrical symmetry. Actually, no source can exist that behaves exactly this way; the real case closest to this situation is that of an indefinitely long straight two-wire line, whose magnetic field — although not cylindrically symmetric — does decay as $1/r^2$, at least starting from a distance sufficiently large compared with the spacing between the pair of conductors.

Case 3 — Magnetic flux density decaying as $1/r^3$ away from the source conductor. In the third case we shall finally assume that the field decays inversely with the cube of the distance away from the source conductor. This situation, too, cannot be physically realised with cylindrical symmetry, but it is useful to represent the way the field decays in the vicinity of many real sources (for instance those confined in a region of space whose maximum linear extent is small compared with the distance at which the field is evaluated).

In all three cases, the quantity constituting the result of the measurement is the potential difference (emf) induced in the loop, equal (in magnitude) to the derivative of the magnetic flux φ linked with the loop itself.

The "ideal" value V_0 of this emf corresponds to the nominal magnetic flux density B_0 at its centre, assumed uniform over the whole loop:

$$B_0 = \frac{k}{d^n} \qquad \text{(Eq. 1)}$$

with n = 1, 2, 3 in the three cases we have agreed to study. The emf V_0 measured under ideal conditions is therefore:

$$V_0 = |\frac{d\varphi_0}{\mathrm{d}t}| = (2\pi f)\, B_0\, (2L)^2 = (2\pi f) \left(\frac{k}{d^n}\right) \left(4L^2\right) = \frac{8\pi f k L^2}{d^n} \qquad \text{(Eq. 2)}$$

The "real" value V of the induced emf is obtained by computing the flux φ as the integral of the actual field value B(x, y) over the area of the loop, i.e. — choosing as the X axis the perpendicular to the source conductor lying on the sensor plane and passing through the sensor centre:

$$B\,(x, y) = \frac{k}{x^n} \qquad \text{(Eq. 3)}$$

Hence, with the integrals taken over the loop area:

$$V = |\frac{d\varphi}{\mathrm{d}t}| = 2\pi f \iint_{\text{loop}} B\,(x, y)\, \mathrm{d}x\mathrm{d}y = 2\pi f \iint_{\text{loop}} \frac{k}{x^n} \mathrm{d}x\mathrm{d}y = 2\pi f k \iint_{\text{loop}} \frac{\mathrm{d}x\mathrm{d}y}{x^n} \qquad \text{(Eq. 4)}$$

that is:

$$V = 2\pi f k \int_{-L}^{+L} \mathrm{d}y \int_{d-L}^{d+L} \frac{\mathrm{d}x}{x^n} = 4\pi f k L \int_{d-L}^{d+L} \frac{\mathrm{d}x}{x^n} \qquad \text{(Eq. 5)}$$

Let us now put:

$$\Gamma_n = \int_{d-L}^{d+L} \frac{\mathrm{d}x}{x^n} \qquad \text{(Eq. 6)}$$

and compute the expression of the relative measurement error ε:

$$\varepsilon = \frac{|V - V_0|}{V_0} = |\frac{V}{V_0} - 1| = |\frac{4\pi f k L \Gamma_n}{\left(\frac{8\pi f k L^2}{d^n}\right)} - 1| = |\frac{d^n}{2L}\Gamma_n - 1| \qquad \text{(Eq. 7)}$$

Case 1 — Magnetic flux density decaying as 1/r

In this case, from Eq. 6 and Eq. 7 with n = 1 we obtain:

$$\Gamma_1 = \int_{d-L}^{d+L} \frac{\mathrm{d}x}{x} = \ln\frac{d+L}{d-L} \qquad \text{(Eq. 8)}$$

and hence:

$$\varepsilon = |\frac{d}{2L}\Gamma_1 - 1| = |\frac{d}{2L}\ln\frac{d+L}{d-L} - 1| \qquad \text{(Eq. 9)}$$

putting ρ = d/L one can write:

$$\varepsilon = |\frac{\rho}{2}\ln\frac{\rho+1}{\rho-1} - 1| \qquad \text{(Eq. 10)}$$

Let us imagine, for instance, dealing with a sensor of about 100 cm² of area, for which therefore 2L = 10 ÷ 12 cm. The error as a function of the distance d of the centre from the source conductor behaves as follows:

Table 1. *Relative measurement error in case 1 (1/r decay).*

d (cm)	ε for L = 5 cm	ε for L = 5.5 cm	ε for L = 6 cm
5.1	135%	N/A	N/A
5.6	61%	140%	N/A
6.1	41%	64%	144%
7	25%	35%	50%
8	17%	23%	30%
9	13%	16%	21%
10	9.9%	12%	16%
12	6.5%	8.0%	9.9%
14	4.6%	5.7%	6.9%
16	3.5%	4.2%	5.1%
18	2.7%	3.3%	4.0%
20	2.2%	2.6%	3.2%

From this table one can see that, to keep the error within 10%, the centre of the sensor must be located at least 10 ÷ 12 cm away from the source conductor (a distance comparable with the side of the sensor itself).

Case 2 — Magnetic flux density decaying as $1/r^2$ (n = 2)

In this case, from Eq. 6 and Eq. 7 with n = 2 we obtain:

$$\Gamma_2 = \int_{d-L}^{d+L} \frac{\mathrm{d}x}{x^2} = \left[-\frac{1}{x}\right]_{d-L}^{d+L} \frac{2L}{d^2 - L^2} \quad \text{(Eq. 11)}$$

and hence:

$$\varepsilon = |\frac{d^2}{2L}\Gamma_2 - 1| = |\frac{d^2}{d^2 - L^2} - 1| = |\frac{L^2}{d^2 - L^2}| \quad \text{(Eq. 12)}$$

putting ρ = d/L one can write:

$$\varepsilon = |\frac{1}{\rho^2 - 1}| \quad \text{(Eq. 13)}$$

With the same assumption as before about the size of the sensor, the error as a function of the distance d of the centre from the source conductor behaves as follows:

Table 2. *Relative measurement error in case 2 ($1/r^2$ decay).*

d (cm)	ε for L = 5 cm	ε for L = 5.5 cm	ε for L = 6 cm
5.1	2475%	N/A	N/A
5.6	393%	2725%	N/A
6.1	205%	435%	2975%
7	104%	161%	277%
8	64%	90%	129%
9	45%	60%	80%
10	33%	43%	56%
12	21%	27%	33%
14	15%	18%	22%
16	11%	13%	16%
18	8.4%	10%	12%
20	6.7%	8.2%	9.9%

The errors, as can be seen, are somewhat larger than in the previous case: to stay below 10% one should place the electrical centre of the sensor at least 18 ÷ 20 cm away from the source conductor (a distance equal to about one and a half times the side of the sensor itself).

Case 3 — Magnetic flux density decaying as $1/r^3$ (n = 3)

In this case, from Eq. 6 and Eq. 7 with n = 3 we obtain:

$$\Gamma_3 = \int_{d-L}^{d+L} \frac{\mathrm{d}x}{x^3} = \left[-\frac{1}{2x^2} \right]_{d-L}^{d+L} \frac{2dL}{(d+L)^2 (d-L)^2} \qquad \text{(Eq. 14)}$$

and hence:

$$\varepsilon = |\frac{d^3}{2L}\Gamma_3 - 1| = |\frac{d^4}{(d+L)^2 (d-L)^2} - 1| \qquad \text{(Eq. 15)}$$

putting ρ = d/L one can write:

$$\varepsilon = |\frac{\rho^4}{(\rho+1)^2 (\rho-1)^2} - 1| = |\frac{\rho^4}{(\rho^2-1)^2} - 1| = |\frac{2\rho^2 - 1}{(\rho^2-1)^2}| \qquad \text{(Eq. 16)}$$

With the same assumption as before about the size of the sensor, the error as a function of the distance d of the centre from the source conductor behaves as follows:

Table 3. *Relative measurement error in case 3 ($1/r^3$ decay).*

d (cm)	ε for L = 5 cm	ε for L = 5.5 cm	ε for L = 6 cm
5.1	66219%	N/A	N/A
5.6	2331%	79719%	N/A
6.1	829%	2758%	94469%
7	317%	583%	1321%
8	169%	260%	422%
9	109%	155%	224%
10	78%	106%	144%
12	46%	60%	78%
14	31%	40%	50%
16	23%	29%	35%
18	17%	22%	27%
20	14%	17%	21%
22	11%	14%	17%
24	9.3%	11%	14%
26	7.8%	9.6%	12%
28	6.7%	8.2%	9.9%
30	5.8%	7.1%	8.5%

In this case, therefore, to stay below 10% one should place the electrical centre of the sensor at least 24 ÷ 28 cm away from the source conductor.

The conclusions reached in the three cases would remain valid even if a circular-loop sensor were considered instead of a square one; in that case, indeed, we can expect a smaller error than in the square-loop case, because the areas of the loop exposed to a field value different from the nominal value B_0 have a smaller extent.

The same conclusions would also remain valid if the sensor did not lie on a plane containing the source conductor; in that case, indeed, the maximum excursion of the field strength from one point of the sensor to another would be smaller than in the case considered (i.e. the sensor would be exposed to a less non-uniform field).

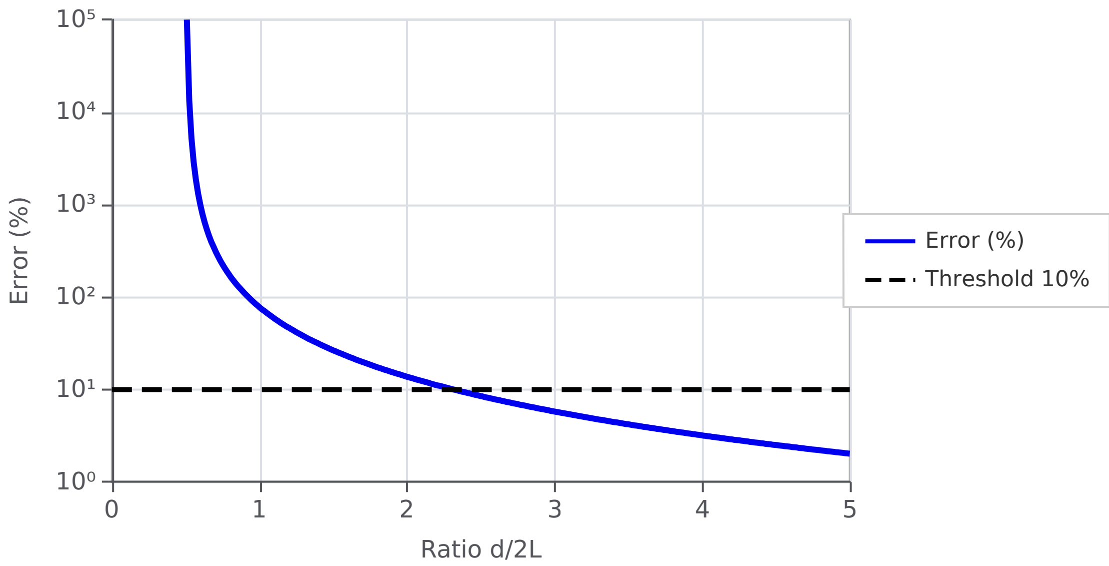

Figure 2. *Relative measurement error in case 3 ($1/r^3$ decay), as a function of the distance d of the sensor centre from the source conductor, normalised to the sensor side 2L.*

The most interesting case from a practical point of view is probably the third one, because it is suitable to represent many real situations and hence to support a rough operational recommendation. Figure 2 refers to it: the estimated error is plotted as a function of the distance d of the sensor centre from the source conductor, normalised to the length 2L of the sensor side. As can be seen, the rough indication that can be given, in order to keep the error within 10%, is to keep the centre of the sensor at a distance from the source equal to at least two and a half times its maximum linear dimension.